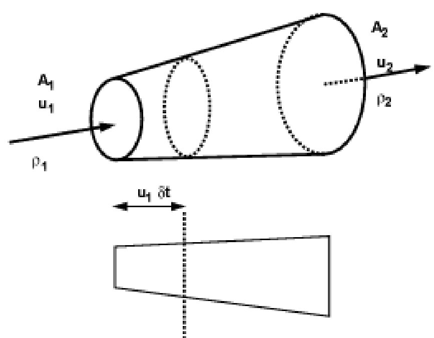
Conservation of Momentum: The Momentum Equation

is a statement of Newton's Second Law. It relates the sum of the forces to the acceleration or rate of change of momentum. From solid mechanics you will recognise

$$F = ma$$

What mass of moving fluid we should use?

We use a different form of the equation. Consider a stream tube and assume steady non-uniform flow:



In time δt a volume of the fluid moves from the inlet a distance $u_1\delta t$, so

volume entering the stream tube = area × distance =$A_1 v_1 \delta t$

mass entering stream tube = volume density = $\rho A_1 v_1 \delta t$

momentum entering stream tube = mass velocity = $\rho A_1 v_1 \delta t\, v_1$

Similarly, at the exit, we get the expression:

momentum leaving stream tube = $\rho A_2 u_2 \delta t\, u_2$

By another reading of Newton's 2nd Law.

where Momentum = m * v

Force = mass x acceleration $= m\dfrac{dv}{dt} = \dfrac{dmv}{dt} =$ rate of change of momentum

$$F = \frac{(\rho_2 A_2 v_2\ \delta t\ v_2 - \rho_1 A_1 v_1 \delta t v_1)}{\delta t}$$

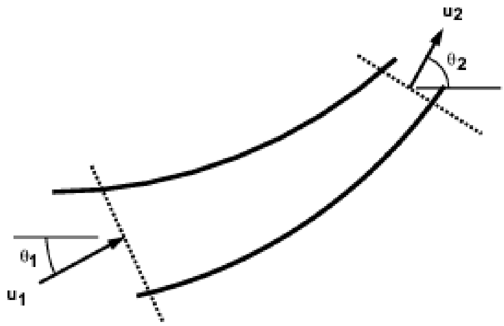
We know from continuity that: $\boldsymbol{Q = A_1 v_1 = A_2 v_2}$

And if we have a fluid of constant density,

$F = Q\rho\ (v_2 - v_1)$

The Momentum equation

This force acts on the fluid in the direction of the flow of the fluid



The previous analysis assumed the inlet and outlet velocities in the same direction (i.e. a one-dimensional system).

What happens when this is not the case?

We consider the forces by resolving in the directions of the co-ordinate axes. The force in the x-direction:

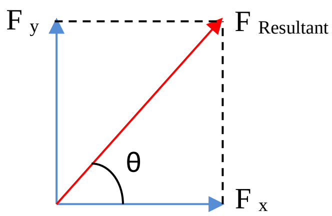
$F_x = \rho Q\ (v_2 \cos\theta_2 - v_1 \cos\theta_1) = \rho Q\ (v_{2x} - v_{1x})$

And the force in the y-direction:

$F_y = \rho Q\ (v_2 \sin\theta_2 - v_1 \sin\theta_1) = \rho Q\ (v_{2y} - v_{1y})$

The resultant force can be found by combining these components

$$F_{Resultant} = \sqrt{F_x{}^2 + F_y{}^2}$$



And the angle of this force :

$$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right)$$

This hydrodynamic force is made up of three components:

F_R = Force exerted on the fluid by any solid body touching the control volume

F_B = Force exerted on the fluid body (e.g. gravity)

F_P = Force exerted on the fluid by fluid pressure outside the control volume

So, we say that the total force, F_T, is given by the sum of these forces:

$$F_T = F_R + F_B + F_P$$

The force exerted by the fluid on the solid body touching the control volume is opposite to F_R ***(action force)***.

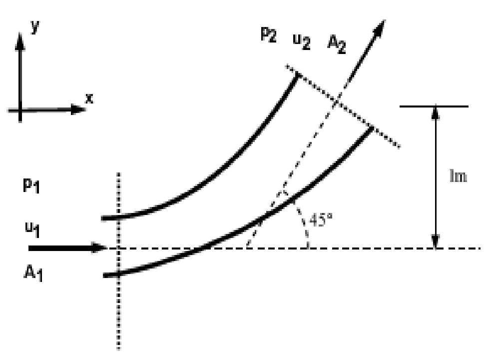
So, the reaction force, R, is given by

$$R = -F_R$$

Application of the Momentum Equation:

Forces on a Bend:

Consider a converging or diverging pipe bend lying in the vertical or horizontal plane turning through an angle of θ.



Here is a diagram of a diverging pipe bend

Why do we want to know the forces here?

As the fluid changes direction, a force will act on the bend.

This force can be very large in the case of water supply pipes. The bend must be held in place to prevent breakage at the joints.

We need to know how much force a support (thrust block) must withstand.

Step in Analysis:

1.Draw a control volume

2.Decide on co-ordinate axis system

3.Calculate the total force

4.Calculate the pressure force

5.Calculate the body force

6.Calculate the resultant force.

Example (1) The outlet pipe from a pump is a bend of 45° rising in the vertical plane (i.e. and internal angle of 135°). The bend is 150 mm diameter at its inlet and 300 mm diameter at its outlet. The pipe axis at the inlet is horizontal and at the outlet it is 1m higher. By neglecting friction, calculate the force and its direction if the inlet pressure is 100 kN/m² and the flow of water through the pipe is 0.3m³/s. The volume of the pipe is 0.075m³.

Solution:

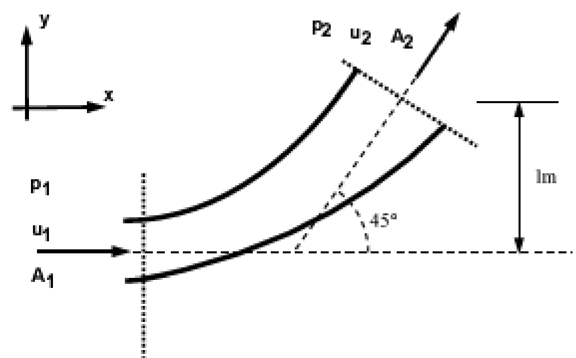
1&2 Draw the control volume and the axis

System

$p_1 = 100\ kN/m^2$,
$Q = 0.3\ m^3/s$
$\theta = 45°$

$d_1 = 0.15$ m $\quad$ $d_2 = 0.3$ m

$A_1 = 0.0177\ m^2$ $\quad$ $A_2 = 0.0707\ m^2$



3. <u>Calculate the total force</u>

in the x direction

$$F_{T_x} = \rho Q(u_{2_x} - u_{1_x}) = \rho Q(u_2 \cos\theta - u_1)$$

by continuity $A_1 u_1 = A_2 u_2 = Q$, so

$$u_1 = \frac{0.3}{\pi\left(0.15^2 / 4\right)} = 16.98\, m/s$$

$$u_2 = \frac{0.3}{0.0707} = 4.24\, m/s$$

$$F_{T_x} = 1000 \times 0.3\left(4.24\cos 45 - 16.98\right)$$
$$= -4193.68\, N$$

and in the y-direction

$$F_{T_y} = \rho Q\left(u_{2_y} - u_{1_y}\right)$$
$$= \rho Q\left(u_2 \sin\theta - 0\right)$$
$$= 1000 \times 0.3(4.24 \sin 45)$$
$$= 899.44\, N$$

4. <u>Calculate the pressure force.</u>

$$F_P = \text{pressure force at 1 - pressure force at 2}$$
$$\theta_1 = 0, \qquad \theta_2 = \theta$$

$$F_{P_x} = p_1 A_1 \cos 0 - p_2 A_2 \cos\theta = p_1 A_1 - p_2 A_2 \cos\theta$$

$$F_{P_y} = p_1 A_1 \sin 0 - p_2 A_2 \sin\theta = -p_2 A_2 \sin\theta$$

We know pressure at the inlet, but not at the outlet, we can use the Bernoulli equation to calculate this unknown pressure.

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

The height of the pipe at the outlet is 1m above the inlet.

Taking the inlet level as the datum:

$z_1 = 0$, $z_2 = 1$m, So the Bernoulli equation becomes:

$$\frac{100000}{1000\times 9.81}+\frac{16.98^2}{2\times 9.81}+0=\frac{p_2}{1000\times 9.81}+\frac{4.24^2}{2\times 9.81}+1.0$$

$$p_2 = 225361.4\ N/m^2$$

$$F_{P_x} = 100000\times 0.0177 - 225361.4\cos 45\times 0.0707$$
$$= 1770 - 11266.34 = -9496.37\ kN$$

$$F_{P_y} = -225361.4\sin 45\times 0.0707$$
$$= -11266.37$$

5. <u>Calculate the body force</u>

The body force is the force due to gravity. That is the weight acting in the negative y-direction.

$$F_{B_y} = -\rho g \times volume$$
$$= -1000\times 9.81\times 0.075$$

$$F_{By} = -735.75N$$

There are no body forces in the x direction,

$$F_{B_x} = 0$$

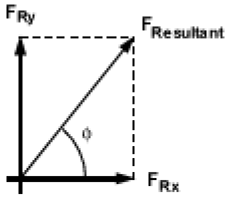
6. <u>Calculate the resultant force</u>

$$F_{T_x} = F_{R_x} + F_{P_x} + F_{B_x}$$
$$F_{T_y} = F_{R_y} + F_{P_y} + F_{B_y}$$

$$F_{R_x} = F_{T_x} - F_{P_x} - F_{B_x}$$
$$= -4193.6 + 9496.37$$
$$= 5302.7\ N$$

$$F_{R_y} = F_{T_y} - F_{P_y} - F_{B_y}$$
$$= 899.44 + 11266.37 + 735.75$$
$$= 12901.56\ N$$

And the resultant force <u>by the fluid</u> is given by:



$$F_R = \sqrt{F_R^2 + F_{R\,y}^2}$$
$$= \sqrt{5302.7^2 + 12901.56^2}$$
$$= 13.95\,kN$$

And the direction of application is

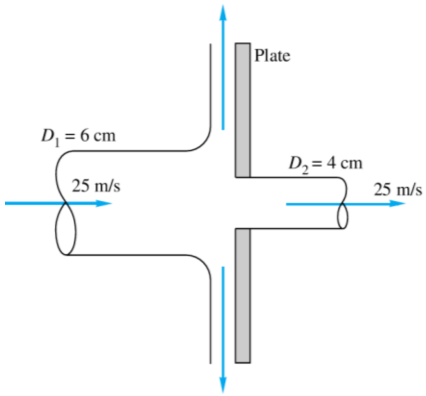
$$\phi = \tan^{-1}\left(\frac{F_{R_y}}{F_{R_x}}\right)$$
$$= \tan^{-1}\left(\frac{12901.56}{5302.7}\right)$$
$$= 67.66^\circ = 67^\circ\, 39'$$

The <u>reaction force by the bend</u> is the same magnitude but in the <u>opposite</u> direction

$$R = -F_R = -13.95\,kN$$

Example (2): The 6-cm-diameter 20°C water jet in Fig. strikes a plate containing a hole of 4-cm diameter. Part of the jet passes through the hole, and part is deflected. Determine the horizontal force required to hold the plat

<u>Solution</u>:



$$Q_{in} = \frac{\pi}{4}(0.06)^2 25 = 0.0707\, m^3/s$$

$$Q_{hole} = \frac{\pi}{4}(0.04)^2 25 = 0.0314\, m^3/s$$

$$F = \rho Q(V_{out} - V_{in})$$

for devided or branched flow

$$F_x = \rho(\sum Q_{out} V_{outx} - \sum Q_{in} V_{inx})$$

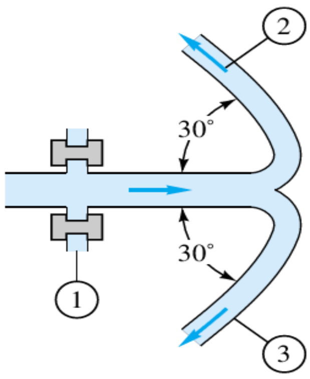
$$F_x = 998(0 + 0 + 0.0314 \times 25 - 0.0707 \times 25)$$

$$F_x = -980\ N$$

$$R_x = 980 N$$

$$F_y = 0$$

Example (3): Water at 20°C exits to the standard sea-level atmosphere through the split nozzle in Fig. Duct areas are $A_1 = 0.02\ m^2$ and $A_2 = A_3 = 0.008\ m^2$. If p1= 135 kPa (absolute) and the flow rate is $Q_2 = Q_3 = 275\ m^3/h$, compute the force on the flange bolts at section 1.



Solution: With the known flow rates, we can compute the various velocities:

$$V_2 = V_3 = \frac{275/3600 \text{ m}^3/\text{s}}{0.008 \text{ m}^2} = 9.55 \frac{\text{m}}{\text{s}}; \quad V_1 = \frac{550/3600}{0.02} = 7.64 \frac{\text{m}}{\text{s}}$$

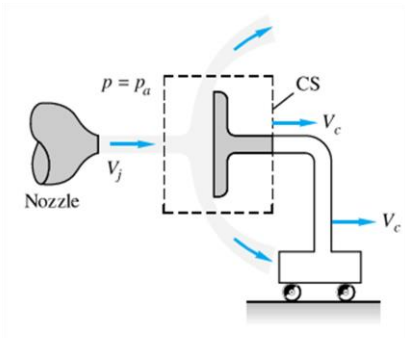
The CV encloses the split nozzle and cuts through the flange. The balance of forces is

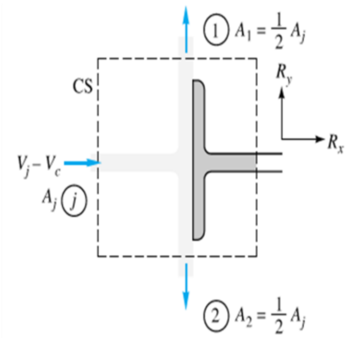
$$\sum F_x = -F_{\text{bolts}} + p_{1,\text{gage}} A_1 = \rho Q_2(-V_2 \cos 30°) + \rho Q_3(-V_3 \cos 30°) - \rho Q_1(+V_1),$$

or: $$F_{\text{bolts}} = 2(998)\left(\frac{275}{3600}\right)(9.55\cos 30°) + 998\left(\frac{550}{3600}\right)(7.64) + (135000 - 101350)(0.02)$$

$$= 1261 + 1165 + 673 \approx \mathbf{3100\ N} \quad \textit{Ans.}$$

Example (4): A water jet of velocity V_j impinges normal to a flat plate which moves to the right at velocity V_c, as shown in Fig. Find the force required to keep the plate moving at constant velocity if the jet density is 1000 kg/m^3, the jet area is 3 cm^2, and V_j and V_c are 20 and 15 m/s, respectively. Neglect the weight of the jet and plate and assume steady flow with respect to the moving plate with the jet splitting into an equal upward and downward half-jet.





Solution:

For moving control volume with $V=V_c$ we have

$$V_{in} = V_j - V_c = 20 - 15 = 5 m/s$$

By continuity equation we have:

$$Q_{in} = Q_{out}$$

$$A_j V_{in} = A_1 V_1 + A_2 V_2 , \; A_1 = A_2 = \frac{1}{2} A_j$$

$$V_{in} = \frac{1}{2} V_1 + \frac{1}{2} V_2 , \quad but \; from \, symmetry \, and \; neglacting \; the \; wight : V_1 = V_2$$

$$V_{in} = V_1 = V_2$$

$$F = \rho Q (V_{out} - V_{in})$$

for devided or branched flow

$$F_x = \rho (\sum Q_{out} V_{outx} - \sum Q_{in} V_{inx})$$

$$F_x = \rho (\sum A_{out} V_{out} V_{outx} - \sum A_{in} V_{in} V_{inx})$$

$$F_x = \rho (0 - 1000 \times 0.0003 \times 5 \times 5)$$

$$F_x = - 7.5 \; N$$

$$F_x = F_{px} + F_{Rx}$$

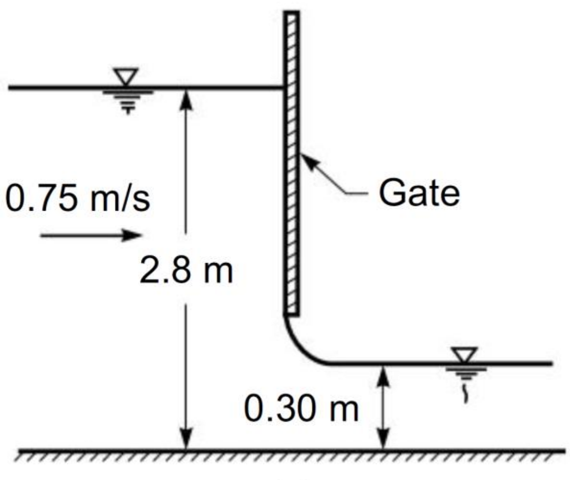
$$F_{Rx} = F_x - F_{px} = -7.5 - 0 = -7.5$$

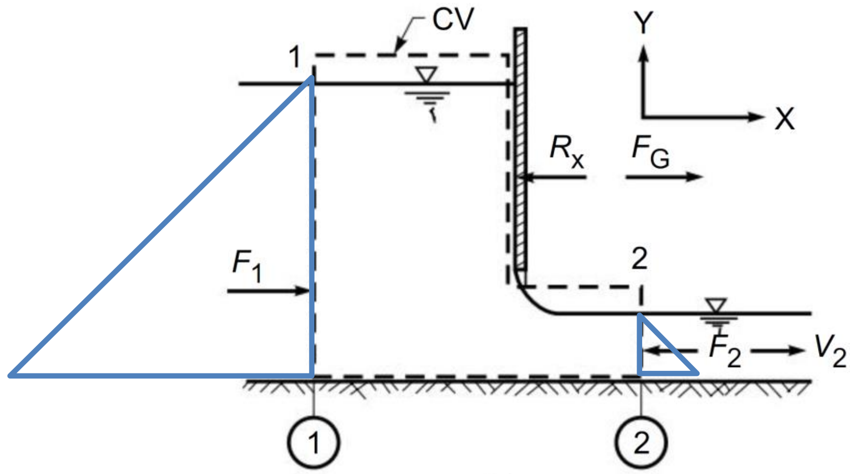
$$R_x = -7.5 N$$

$$F_y = 0$$

Example (5):

A sluice gate in an open channel is shown in Fig. 5.20(a). Estimate the force on the unit width of the gate. Neglect frictional force on the channel bottom.





Solution: Consider a unit width of the gate and the control volume (*cv*) as shown in Fig. 5.20(b). The frictional force on the channel bottom is neglected.

The forces on the control volume surfaces are:

F_1 = pressure force on the section 1 = $\gamma y_1^2/2$ (by assuming hydrostatic pressure distribution)

F_2 = pressure force on the section 2 = $\gamma y_2^2/2$

R_x = reaction of the gate on the water in the control volume acting in the (– x)-direction.

Here

$y_1 = 2.8$ m, $y_2 = 0.3$ m, $V_1 = 0.75$ m/s

q = discharge per unit width of channel

$= y_1V_1 = y_2V_2 = 0.75 \times 2.8 = 2.1$ m^3/s/m

$V_2 = q/y_2 = 2.1/0.3 = 7.0$ m/s

From momentum equation to the control volume in the x-direction.

$$F_1 - F_2 - R_x = \rho Q(V_2 - V_1)$$

$$\frac{1}{2} \times 9.79 \times (2.8)^2 - \frac{1}{2} \times 9.79 \times (0.3)^2 - R_x$$

$$= \frac{998}{1000}\ (2.1)\ (7.0 - 0.75)$$

$$38.38 - 0.44 - R_x = 13.10$$

$$R_x = 24.84 \text{ kN}$$

The force F_G on the gate due to water is equal and opposite to R_x. **Thus F_G = 24.84 kN and acts in the positive x-direction**.